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3 (Sem-2/CBCS) MAT HC 1

2022

MATHEMATICS

(Honours)

Paper : MAT-HC-2016

(Real Analysis)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer **any ten** questions : $1 \times 10 = 10$

(a) Find the infimum of the set

$$\left\{ 1 - \frac{(-1)^n}{n} : n \in \mathbb{N} \right\}$$

(b) If A and B are two bounded subsets of \mathbb{R} , then which one of the following is true?

(i) $\sup(A \cup B) = \sup\{\sup A, \sup B\}$

(ii) $\sup(A \cup B) = \sup A + \sup B$

Contd.

- (iii) $\sup(A \cup B) = \sup A \cdot \sup B$
- (iv) $\sup(A \cup B) = \sup A \cup \sup B$
- (c) There does not exist a rational number x such that $x^2 = 2$. (Write True or False)
- (d) The set Q of rational numbers is uncountable. (Write True or False)
- (e) If $I_n = \left(0, \frac{1}{n}\right)$ for $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} I_n = ?$
- (f) The convergence of $\{|x_n|\}$ imply the convergence of $\{x_n\}$. (Write True or False)
- (g) What are the limit points of the sequence $\{x_n\}$, where $x_n = 2 + (-1)^n$, $n \in \mathbb{N}$?
- (h) If $\{x_n\}$ is an unbounded sequence, then there exists a properly divergent subsequence. (Write True or False)
- (i) A convergent sequence of real numbers is a Cauchy sequence. (Write True or False)

(j) If $0 < a < 1$ then $\lim_{n \rightarrow \infty} a^n = ?$

(k) The positive term series $\sum \frac{1}{n^p}$ is convergent if and only if

(i) $p > 0$

(ii) $p > 1$

(iii) $0 < p < 1$

(iv) $p \leq 1$

(Write correct one)

(l) Define conditionally convergent of a series.

(m) If $\{x_n\}$ is a convergent monotone

sequence and the series $\sum_{n=1}^{\infty} y_n$ is

convergent, then the series $\sum_{n=1}^{\infty} x_n y_n$ is

also convergent.

(Write True or False)

(n) Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^m \left(1 + \frac{1}{n^p}\right)}$

where m and p are real numbers under which of the following conditions does the above series convergent ?

(i) $m > 1$

(ii) $0 < m < 1$ and $p > 1$

(iii) $0 \leq m \leq 1$ and $0 \leq p \leq 1$

(iv) $m = 1$ and $p > 1$

(o) Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers defined by $x_1 = 1$, $y_1 = \frac{1}{2}$,

$$x_{n+1} = \frac{x_n + y_n}{2} \text{ and } y_{n+1} = \sqrt{x_n y_n} \quad \forall n \in \mathbb{N}$$

then which one of the following is true ?

(i) $\{x_n\}$ is convergent, but $\{y_n\}$ is not convergent

(ii) $\{x_n\}$ is not convergent, but $\{y_n\}$ is convergent

(iii) Both $\{x_n\}$ and $\{y_n\}$ are convergent and $\lim_{n \rightarrow \infty} x_n > \lim_{n \rightarrow \infty} y_n$

(iv) Both $\{x_n\}$ and $\{y_n\}$ are convergent and $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n$

2. Answer **any five** parts : $2 \times 5 = 10$

(a) If a and b are real numbers and if $a < b$, then show that $a < \frac{1}{2}(a + b) < b$.

(b) Show that the sequence $\left\{ \frac{2n-7}{3n+2} \right\}$ is bounded.

(c) If $\{x_n\}$ converges in \mathbb{R} , then show that $\lim_{n \rightarrow \infty} x_n = 0$

(d) Show that the series $1+2+3+\dots$, is not convergent.

(e) Test the convergence of the series :

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

(f) State Cauchy's integral test of convergence.

(g) State the completeness property of \mathbb{R} and

find the $\sup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$.

(h) Does the Nested Interval theorem hold for open intervals? Justify with a counter example.

3. Answer **any four** parts : $5 \times 4 = 20$

(a) If x and y are real numbers with $x < y$, then prove that there exists a rational number r such that $x < r < y$.

(b) Show that a convergent sequence of real numbers is bounded.

(c) Prove that $\lim_{n \rightarrow \infty} \left(n^{\frac{1}{n}} \right) = 1$.

(d) $\{x_n\}$ be a sequence of real numbers that converges to x and suppose that $x_n \geq 0$. Show that the sequence $\{\sqrt{x_n}\}$ of positive square roots converges and

$$\lim_{n \rightarrow \infty} \sqrt{x_n} = \sqrt{x}.$$

(e) Show that every absolutely convergent series is convergent. Is the converse true? Justify. $4+1=5$

(f) Using comparison test, show that the series $\sum \left(\sqrt{n^4+1} - \sqrt{n^4-1} \right)$ is convergent.

(g) State Cauchy's root test. Using it, test the convergence of the series

$$\left(\frac{2^2}{1^2} - \frac{2}{1} \right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2} \right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3} \right)^{-3} + \dots$$

$1+4=5$

(h) Show that the sequence defined by the recursion formula

$$u_{n+1} = \sqrt{3u_n}, \quad u_1 = 1$$

is monotonically increasing and bounded. Is the sequence convergent? $2+2+1=5$

4. Answer **any four** parts : $10 \times 4 = 40$

(a) Show that the sequence $\left\{ \left(1 + \frac{1}{n} \right)^n \right\}$ is

convergent and $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e$ which lies between 2 and 3.

(b) (i) Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are sequences of real numbers such that $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n.$$

Show that $\{y_n\}$ is convergent and

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} z_n \quad 5$$

(ii) What is an alternating series? State Leibnitz's test for alternating series. Prove that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \infty$ is a conditionally convergent series. $1+1+3=5$

(c) Test the convergence of the series

$$1 + a + a^2 + \dots + a^n + \dots$$

(d) (i) Using Cauchy's condensation test, discuss the convergence of the

$$\text{series } \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} \quad 5$$

(ii) Define Cauchy sequence of real numbers. Show that the sequence

$$\left\{ \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} \right\} \quad \text{is a}$$

Cauchy sequence. $1+4=5$

(e) (i) Show that a convergent sequence of real numbers is a Cauchy sequence. 5

(ii) Using Cauchy's general principle of convergence, show that the

sequence $\left\{ 1 + \frac{1}{2} + \dots + \frac{1}{n} \right\}$ is not convergent. 5

(f) (i) Prove that every monotonically increasing sequence which is bounded above converges to its least upper bound. 5

(ii) Show that the limit if exists of a convergent sequence is unique.

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(g) State and prove p -series.

(h) (i) Test the convergence of the series

$$x + \frac{3}{5}x^2 + \frac{8}{10}x^3 + \dots + \frac{n^2 - 1}{n^2 + 1}x^n + \dots \quad (x > 0)$$

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(ii) If $\{x_n\}$ is a bounded increasing sequence then show that

$$\lim_{n \rightarrow \infty} x_n = \sup\{x_n\} \quad 5$$

(i) (i) Show that a bounded sequence of real numbers has a convergent subsequence. 5

(ii) State and prove Nested Interval theorem. 5

(j) (i) Show that Cauchy sequence of real numbers is bounded. 5

(ii) Test the convergence of the series

$$x^2 + \frac{2^2}{3.4}x^4 + \frac{2^2.4^2}{3.4.5.6}x^6 + \frac{2^2.4^2.6^2}{3.4.5.6.7.8}x^8 + \dots \quad (x > 0)$$

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