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3 (Sem-4/CBCS) MAT HC 1

2022

MATHEMATICS

(Honours)

Paper : MAT-HC-4016

(Multivariate Calculus)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer **any ten** : 1×10=10

(i) Find the domain of $f(x, y) = \frac{1}{\sqrt{x-y}}$.

(ii) How is directional derivative of a function at a point related to the gradient of the function at that point?

(iii) Define harmonic function?

(iv) Define $\iint_R f(x, y) dA$.

Contd.

- (v) Write the value of $\bar{\nabla}(f^n)$.
- (vi) Define critical point.
- (vii) Define relative extrema for a function of two variables.
- (viii) When is a curve said to be positively oriented?
- (ix) Describe the fundamental theorem of line integral.
- (x) When is a surface said to be smooth?
- (xi) Compute $\int_1^4 \int_{-2}^3 \int_2^5 dx dy dz$.
- (xii) Evaluate $\lim_{(x,y) \rightarrow (1,3)} \frac{x-y}{x+y}$.
- (xiii) If $f(x, y) = x^3y + x^2y^2$, find f_x .
- (xiv) When is a line integral said to be path independent?
- (xv) Explain the difference between $\int_C f ds$ and $\int_C f dx$.

2. Answer **any five** questions : $2 \times 5 = 10$
- (a) Sketch the level surface $f(x, y, z) = c$ if $(x, y, z) = y^2 + z^2$ for $c = 1$.
- (b) Determine f_x and f_y for $f(x, y) = xy^2 \ln(x+y)$.
- (c) Find $\frac{\partial w}{\partial t}$ if $w = \ln(x + 2y - z^2)$ and $x = 2t - 1, y = \frac{1}{t}, z = \sqrt{t}$.
- (d) Evaluate $\int_1^2 \int_0^\pi x \cos y dy dx$.
- (e) Evaluate $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + x - xy - y}{x - y}$.
- (f) Define line integral over a smooth curve.
- (g) Find the Jacobian $\frac{\partial(x, y)}{\partial(u, v)}$ when $x = u + 2v, y = 3u - 4v$.
- (h) Using polar coordinates find the limit $\lim_{(x,y) \rightarrow (0,0)} \frac{\tan(x^2 + y^2)}{x^2 + y^2}$.

3. Answer **any four**: 5×4=20

(a) Describe the graph of the function

$$f(x, y) = 1 - x - \frac{1}{2}y.$$

(b) Use the method of Lagrange's multipliers to find the maximum and minimum values of $f(x, y) = 1 - x^2 - y^2$ subject to the constraints $x + y = 1$ with $x \geq 0, y \geq 0$.

(c) Evaluate $\int_C [(y - x)dx + x^2 y dy]$, where C is the curve defined by $y^2 = x^3$ from $(1, -1)$ to $(1, 1)$.

(d) Examine the continuity of the following function at the origin:

$$f(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

(e) Find $\frac{\partial w}{\partial s}$ if $w = 4x + y^2 + z^3$ where

$$x = e^{rs^2}, \quad y = \ln \frac{r+s}{t} \quad \text{and} \quad z = rst^2.$$

(f) Suppose the function f is differentiable at the point P_0 and that the gradient at P_0 satisfies $\Delta f_0 \neq 0$. Show that Δf_0 is orthogonal to the level surface of f through P_0 .

(g) Compute $\iint_D \left(\frac{x-y}{x+y} \right)^4 dy dx$ where D is the triangular region bounded by the line $x + y = 1$ and the coordinate axes, using change of variables $u = x - y, v = x + y$.

(h) Find the absolute extrema of $f(x, y) = 2x^2 - y^2$ on the closed bounded set S , where S is the disk $x^2 + y^2 \leq 1$.

4. Answer **any four** questions: 10×4=40

(a) The radius and height of a right circular cone are measured with errors of at most 3% and 2% respectively. Use increments to approximate the maximum possible percentage error in computing the volume of the cone using these measurements and the formula $V = \frac{1}{3}\pi R^2 H$.

(b) Let $f(x, y) = \begin{cases} xy \left(\frac{x^2 - y^2}{x^2 + y^2} \right), & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$

Show that $f_x(0, y) = -y$ and $f_x(x, 0) = x$ for all x and y . Then show that $f_{xy}(0, 0) = -1$ and $f_{yx}(0, 0) = 1$.

(c) (i) Find the directional derivative of $f(x, y) = \ln(x^2 + y^3)$ at $P_0(1, -3)$ in the direction of $\vec{v} = 2i - 3j$.

(ii) In what direction is the function defined by $f(x, y) = xe^{2y-x}$ increasing most rapidly at the point $P_0(2, 1)$, and what is the maximum rate of increase? In what direction is f decreasing most rapidly?

(d) When two resistances R_1 and R_2 are connected in parallel, the total resistance R satisfies $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If R_1 is measured as 300 ohms with maximum error of 2% and R_2 is measured as 500 ohms with a maximum error of 3%, what is the maximum percentage error in R ?

(e) Verify the vector field $\vec{F} = (e^x \sin y - y)i + (e^x \cos y - x - 2)j$ is conservative. Also find the scalar potential function f for \vec{F} .

(f) (i) Evaluate $\iiint_D \frac{dx dy dz}{\sqrt{x^2 + y^2 + z^2}}$ where D is the solid sphere $x^2 + y^2 + z^2 \leq 3$.

(ii) Find the volume of the solid D , where D is bounded by the paraboloid $z = 1 - 4(x^2 + y^2)$ the xy -plane.

(g) (i) Use a polar double integral to show that a sphere of radius a has volume $\frac{4}{3}\pi a^3$.

(ii) Evaluate $\int_0^3 \int_0^{\sqrt{9-x^2}} x dy dx$ by converting to polar coordinates.

(h) State Green's theorem. Verify Green's theorem for the line integral $\oint_C (y^2 dx + x^2 dy)$ where C is the square having vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$.

(i) State Stokes' theorem. Using Stokes' theorem evaluate the line integral $\oint_C (x^3 y^2 dx + dy + z^2 dz)$, where C is the circle $x^2 + y^2 = 1$ and in the plane $z = 1$, counterclockwise when viewed from the origin.

(j) A container in R^3 has the shape of the cube given by $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$. A plate is placed in the container in such a way that it occupies that portion of the plane $x + y + z = 1$ that lies in the cubical container. If the container is heated so that the temperature at each point (x, y, z) is given by $T(x, y, z) = 4 - 2x^2 - y^2 - z^2$ in hundreds of degrees Celsius, what are the hottest and coldest points on the plate? You may assume these extreme temperatures exist.