*Total number of printed pages–6*

**3 (Sem–4 /CBCS) MAT HC 3**

**2021**

**MATHEMATICS**

( Honours )

Paper : MAT–HC–4036

***( Ring Theory )***

*Full Marks : 80*

Time : Three hours

***The figures in the margin indicate full marks for the questions.***

1. $1\times6=6$

(a) Give an example of an infinite non-commutative ring with unity.

(b) What is the characteristic of the ring $Z_3[i]$ ?

(c) Find all the idempotent elements of $Z_{10}$.

(d) Let $f(x)=4x^3+2x^2+x+3$ and $g(x)=3x^4+3x^3+3x^2+x+4$ where $f(x),\ g(x)\in Z_5[x]$. Compute $f(x).g(x)$.

*Contd.*

(e) Show that the polynomial $x^5+9x^4+12x^2+6$ is irreducible over $Q$.

(f) Define an Euclidean Domain.

2. 2×5=10

(a) Prove that in a ring $R$, $a(-b)=(-a)b=-(ab)$ for all $a$, $b \in R$.

(b) Let $A$ and $B$ be two ideals of a ring $R$. Show that $AB \subseteq A \cap B$.

(c) If $A$ is an ideal of a ring $R$ and $1 \in A$, then prove that $A = R$.

(d) If $R$ is a commutative ring with unity and $A$ is an ideal of $R$, show that $R/A$ is a commutative ring with unity.

(e) Let $f(x) = x^3+2x+4$ and $g(x)=3x+2$ in $Z_5(x)$. Determine the quotient and remainder upon dividing $f(x)$ by $g(x)$.

3. Answer ***any four*** parts : 6×4=24

(a) Let $R$ be a commutative ring with unity and $A$ be an ideal of $R$. Prove that $R/A$ is a field if and only if $A$ is a maximal ideal. 6

(b) (i) Let $R$ be a commutative ring and $A$ be an ideal of it. Show that the set $\{r \in R | ra = 0, \text{ for all } a \in A\}$ is an ideal of $R$. 3

(ii) Prove that the characteristic of an integral domain is $0$ or prime. 3

(c) Define a principal ideal domain. If $F$ is a field, then show that $F(x)$ is a principal ideal domain. 1+5=6

(d) Let $p$ be a prime and $f(x) \in Z(x)$ with $deg\ f(x) \geq 1$. Suppose $\overline{f(x)}$ be the polynomial in $Z_p[x]$ obtained from $f(x)$ by reducing all the coefficients of $f(x)$ modulo $p$. If $\overline{f(x)}$ is irreducible over over $Z_p$ and $deg\ f(x) = deg\ \overline{f(x)}$, then prove that $f(x)$ is irreducible over $Q$.

Is the converse true ? Justify your answer. 4+2=6

(e) Prove that in a principal ideal domain, an element is an irreducible if and only if it is a prime. 6

(f) (i) Prove that the ring of integers $Z$ is an Euclidean Domain. 2

(ii) Prove that every Euclidean Domain is a principal ideal domain. 4

4. Answer ***any four*** parts : 10×4=40

(a) (i) Let $Z[\sqrt{2}] = \{a+b\sqrt{2} \mid a, b \in Z\}$. Prove that $Z(\sqrt{2})$ is a ring under the ordinary addition and multiplication of real numbers. 6

(ii) Let $R$ be a ring. Prove that $a^2 - b^2 = (a+b)(a-b)$ for all $a$, $b$ in $R$ if and only if $R$ is commutative. 4

(b) (i) Define a field. Prove that a finite integral domain is a field. Hence show that for any prime $p$, $Z_p$ the ring of integers modulo $p$, is a field. 1+5+2=8

(ii) Show that *0* is the only nilpotent element in an integral domain. 2

(c) (i) Show that $S = \{a+ib \mid a, b \in Z,\ b \text{ is even}\}$ is a subring of $Z[i]$ but not an ideal of $Z[i]$. 4

(ii) Prove that the only ideals of a field $F$ are $\{0\}$ and $F$ itself. 2

(iii) Show that $R[x] / \langle x^2+1 \rangle$ is a field. 4

(d) (i) Let $\phi$ be a monomorphism from a ring $R$ to a ring $S$.
Prove that kernel of $\phi$ is an ideal of $R$. 4

(ii) Let $\phi$ be a homomorphism from a ring $R$ to a ring $S$. Prove that $R / Ker\,\varphi \cong \phi(R)$. 6

(e) (i) If $\phi$ is an isomorphism from a ring $R$ to a ring $S$, then show that $\phi^{-1}$ is an isomorphism from $S$ to $R$. 5

(ii) Let $R$ be a ring with unity $e$. Prove that the mapping $\phi : Z \to R$ given by $n \to ne$ is a ring homomorphism. 5

(f) Let $F$ be a field and let $f(x)$ and $g(x) \in F[x]$ with $g(x) \neq 0$. Prove that there exist unique polynomials $q(x)$ and $r(x)$ in $F(x)$ such that $f(x) = g(x)q(x) + r(x)$ and either $r(x) = 0$ or $deg\, r(z) < deg\, g(x)$.

Hence show that if $a \in F$ and $f(x) \in F[x]$, then $f(a)$ is the remainder in the division of $f(x)$ by $x - a$.

7+3=10