Schrodinger time independent wave equation:

From the classical mechanics, the amplitude of a particle wave moving along x axis can be written as

$$\Psi_{(x)} = 2A\sin(2\pi x/\lambda)$$

Differentiating the wave function w.r.t x,

$$\frac{dy}{dx}\Psi_{(x)} = 2A\cos(2\pi x/\lambda)\ \times 2\pi/\lambda$$

$$\Rightarrow \frac{d^2}{dx^2}\Psi_{(x)} = 2A\,(-\sin(2\pi x/\lambda))\ \times (2\pi/\lambda)^2$$

$$\Rightarrow \frac{d^2}{dx^2}\Psi_{(x)} = -\frac{4\pi^2}{\lambda^2}\Psi_{(x)}$$

For a microparticle, according to de Broglie equation,

$$\lambda = \frac{h}{mv}$$

So,

$$\frac{d^2}{dx^2}\Psi_{(x)} = -\frac{4m^2v^2\pi^2}{h^2}\Psi_{(x)}$$

The total energy(E) of the particle is the sum of kinetic energy $1/2mv^2$ and potential energy V. Therefore,

$$E=\frac{1}{2}mv^2 + V$$

$\Rightarrow m^2v^2 = 2m(E-V)$

Substituting this to above equation,

$$\frac{d^2}{dx^2}\Psi_{(x)} = -\frac{8m\pi^2}{h^2}(E-V)\Psi_{(x)}$$

$$\Rightarrow \frac{d^2}{dx^2}\Psi_{(x)} + \frac{8m\pi^2}{h^2}(E-V)\Psi_{(x)} = 0$$

This is the Schrodinger equation for a single particle of mass m moving in x direction.
For three dimensions,

$$\frac{d^2}{dx^2}\Psi + \frac{d^2}{dy^2}\Psi + \frac{d^2}{dz^2}\Psi + \frac{8m\pi^2}{h^2}(E-V)\Psi = 0$$

Where ψ and V are functions of coordinates x,y, and z and ψ is known as wave function.
The above equation can be rearranged as

$$\frac{h^2}{8\pi^2 m}(\frac{d^2}{dx^2}\Psi + \frac{d^2}{dy^2}\Psi + \frac{d^2}{dz^2}\Psi) + V)\Psi = -E\Psi$$

$$\Rightarrow [\frac{-h^2}{8\pi^2 m}(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2})+V]\psi = E\psi$$

$$\Rightarrow \hat{H}\psi = E\psi$$

Where

$$\hat{H} = \frac{-h^2}{8\pi^2 m}(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2})+V$$

$$\Rightarrow \hat{H} = \frac{-h^2}{8\pi^2 m}\nabla^2 + V$$

is called Hamiltonian operator and

$$\nabla^2 = \frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$$

Condition for a physically accepted, well behaved, realistic wave function:

1. $\psi_{(x)}$ should be finite, single-valued and continuous everywhere in space.

2. $\frac{dy}{dx}\psi_{(x)}$ should be continuous everywhere in space.

3. $\psi_{(x)}$ should be square integrable i.e. $\int_{-\infty}^{\infty}\left|\psi_{(x)}\right|^2 dx = finite\ number.$

Physical significance of wave function

Generally, ψ (x) is a complex quantity. It can be multiplied by any complex number without affecting its physical significance. In general, ψ (x) has no direct physical significance. But the quantity $\psi^*_{(x)}\psi_{(x)}$ is real, physically significant and is defined as position probability density i.e. probability of finding the particle per unit length at time ‘t’. Therefore, for 1-D motion the probability of finding the particle between x to x + dx at time ‘t’ is given by

$\psi^*_{(x)}\psi_{(x)}dx = \left|\psi_{(x)}\right|^2 dx$

Now, $\psi_{(x)}$ should be chosen such that total probability of finding the particle in the entire space should be equal to unity i.e.

$$\int \psi^*_{(x)}\,\psi_{(x)}dx = 1$$

This is called the normalization condition of the wave function.

Condition for Normalization: Normalization constant

If $\varphi_{(x)}$ is unnormalized wave function and if $\psi_{(x)} = N\varphi(x)$, then N is normalization constant if

$$\int \psi *_{(x)} \psi_{(x)} dx = 1$$

$$\Rightarrow \int N\varphi *_{(x)} N\varphi_{(x)} dx = 1$$

$$\Rightarrow N^2 \int \varphi *_{(x)} \varphi_{(x)} dx = 1$$

$$\Rightarrow N^2 = \frac{1}{\int \varphi *_{(x)} \varphi_{(x)} dx}$$

$$\Rightarrow N = \sqrt{\frac{1}{\int \varphi *_{(x)} \varphi_{(x)} dx}}$$